PROBABILITY BOARDWORK (PART 1)

- Recall – when we have a random process, there are many possible outcomes that can occur
  - e.g. – flip a coin. A possible outcome is Heads. Another outcome is Tails
  - roll a die. A possible outcome is 1, 2, ..., 6.
- sample space is the set of possible outcomes
- A random variable (r.v.) is a function from a sample space to the real numbers
  - often denoted with a capital letter like X or Y
- we have two broad classes of r.v.'s: discrete + continuous

Discrete Random Variables

- a random variable is discrete if it takes on a finite or countably infinite number of values

e.g. presence/absence, # of fish

  - i.e. there is a list of values $\{a_1, a_2, \ldots\}$ for which $Pr(X = a_i) > 0$ for all i

"probability that r.v. X is equal to a_i"

  - this list of values (for which probability is positive) is called the support of the r.v. X. Denote as S_X.

- the distribution (how the r.v. behaves) of discrete r.v. X is characterized by its probability mass function (PMF)

Notation (all equivalent): $f_X(x) = Pr(\underbrace{X = x}_{\text{event}}) = [x]$

r.v. specific value

- the PMF specifies the probability of all events associated with r.v. X

- The PMF has two properties:
  1) $f_X(x) = Pr(X = x) = [x] \geq 0$ for all x
     - interpretation: prob. that X equals any value is non-negative

2) $\sum_{x \in S_X} Pr(X=x) = \sum_{x \in S_X} [x] = 1$

- interpretation: prob. must sum to 1

e.g. suppose we are flipping a fair coin 3 times.
Let X be the r.v. for the # of Heads from these three flips
$\Rightarrow S_X = \{0, 1, 2, 3\}$

Q: what is PMF of X?

A: simply need to obtain prob. of each value in S_X.

x	0	1	2	3
$[x]$	?			

$Pr(X=0) = [0] = \frac{\text{\# ways observe } X=0}{\text{Total \# of possible outcomes}} = \frac{1}{8}$

Possible outcomes: HHH HTH THT
HHT TTH THH
HTT TTT

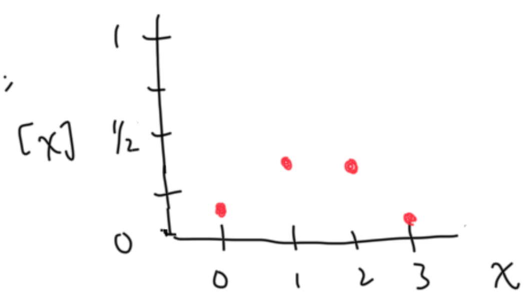
$Pr(X=1) = [1] = \frac{3}{8}$

$\Rightarrow$

x	0	1	2	3
$[x]$	$1/8$	$3/8$	$3/8$	$1/8$

verify that the two properties are satisfied!

Visualize the PMF:



- we have discussed probability of event $X = x$.
- what about event $X \leq x$?

- The cumulative distribution function (CDF) of r.v. X is defined for any value a as:

$$F_X(a) = Pr(X \leq a)$$

- if X is discrete, then $Pr(X \le a) = \sum_{\substack{y \le a, \\ y \in S_X}} Pr(X = y)$

sum over all outcomes y for which $y \le a$!

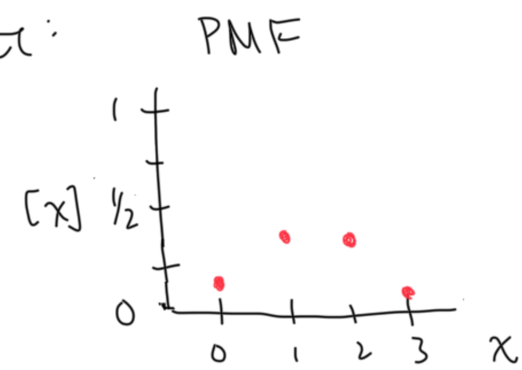
e.g. cont. $F_X(-1) = Pr(X \le -1) = \sum_{\substack{y \le -1 \\ y \in S_X}} [y] = 0$

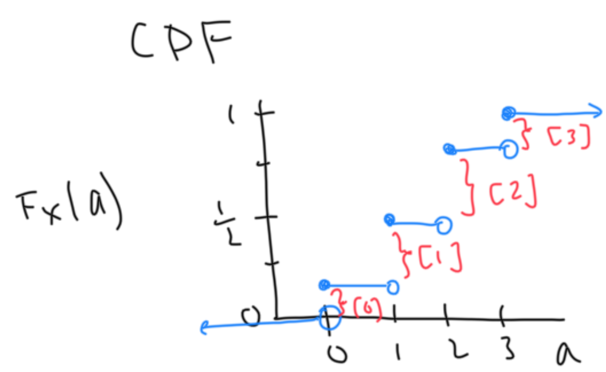
$F_X(0) = Pr(X \le 0) = \sum_{\substack{y \le 0, \\ y \in S_X}} [y] = [0] = \frac{1}{8}$

$F_X(\frac{1}{2}) = Pr(X \le 0) = \sum_{\substack{y \le 0, \\ y \in S_X}} [y] = [0] = \frac{1}{8}$

$F_X(4) = Pr(X \le 4) = \quad \ldots \quad = [0] + [1] + [2] + [3] = 1$

visualize:





Note: CDF for discrete X is a step function; the size of steps are $[x]$ for all $x \in S_X$

Properties of CDF:
1) Non-decreasing
2) Reaches 0 as we move towards $-\infty$
3) Reaches 1 as we move towards $+\infty$

- Great! We have a way to talk about behavior of a discrete r.v. But what if I don't want to bring out the PMF? Is there a way we can obtain some single value that describes X?
  - One way is to talk about the moments of X, which can be used to obtain summary measures about shape of dist.
    - common ones are mean + variance

- If X discrete:
  - the expected value of X is the value

$$\mathbb{E}[X] = \mu_X = \sum_{x \in S_X} x \Pr[X = x] = \sum_{x \in S_X} x[x]$$

- "typical" value of X; weighted average of values in support

- in general, $\mathbb{E}[g(X)] = \sum_{x \in S_X} g(x)[x]$
  - weighted average of $g(x)$, considering all possible values that $g(x)$ can be

- the variance of X is the value

always non-negative. →

$$\text{var}(X) = \sigma_X^2 = \mathbb{E}[(X - \mu_X)^2] = \sum_{x \in S_X} (x - \mu_X)^2 [x]$$

- measure of spread about the mean
- note: standard deviation σ_X is simply the squareroot of variance:

$$\sigma_X = \sqrt{\sigma_X^2}$$

- eg. cont. - Find expected value & variance of X

$$\mathbb{E}[X] = \mu_X = \sum_{x \in S_X} x[x] = 0[0] + 1[1] + 2[2] + 3[3]$$

$$= 0\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\left(\tfrac{3}{8}\right) = 3\left(\tfrac{1}{8}\right)$$

$$= 1.5$$

- this makes sense! Also, note $\mu_X \notin S_X$

$$\text{var}(X) = \sigma_X^2 = \sum_{x \in S_X} (x - 1.5)^2 [x]$$

$$= (0-1.5)^2[0] + (1-1.5)^2[1] + (2-1.5)^2[2] + (3-1.5)^2[3]$$

$$= \tfrac{3}{4}$$