Likelihood

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Overview

- Generative data model
- Stochastic and parametric models
- Decisions
  - Support
  - Characteristics
  - Complexity
  - Dependence
- Likelihood functions
- MLEs and properties

Data Model

$$y_i \sim [y_i|\boldsymbol{\theta}] \text{ , for } i = 1, \ldots, n$$

- y_i: observation i
- $\boldsymbol{\theta}$: parameter(s)
- n: number of observations (sample size)
- i: observation index
- $\sim$: "distributed as"
- $[y_i|\boldsymbol{\theta}]$: probability distribution of y_i given $\boldsymbol{\theta}$

Fish Survival

- n: fish mesocosms (tanks).
- N: initial number of fish put into each tank.
- y_i: number of fish survived in tank i after 24 hrs.
- ϕ: survival probability.

Generative Parametric Models

- Binomial:

  $y_i \sim \text{Binom}(N, \phi)$, for $i = 1, \ldots, n$

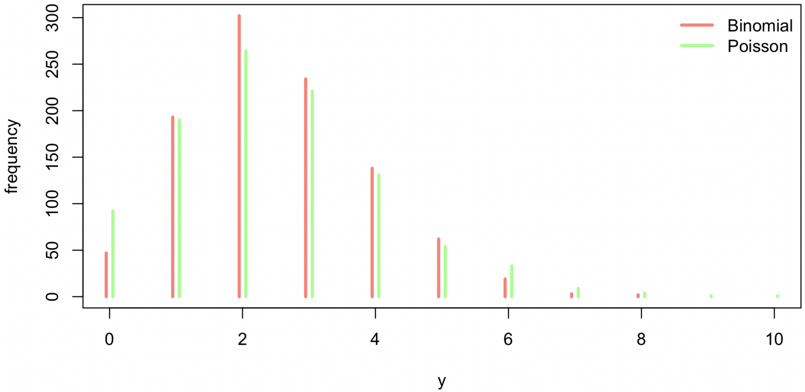
  - $\text{E}(y_i|N, \phi) = N \cdot \phi$
  - $\text{Var}(y_i|N, \phi) = N \cdot \phi \cdot (1 - \phi)$

- Poisson:

  $y_i \sim \text{Pois}(N \cdot \phi)$, for $i = 1, \ldots, n$

  - $\text{E}(y_i|N, \phi) = N \cdot \phi$
  - $\text{Var}(y_i|N, \phi) = N \cdot \phi$

$N = 10, \phi = 0.25, n = 1000$



Aspects

- Binomial:

  $y_i \sim \text{Binom}(N, \phi)$, for $i = 1, \ldots, n$
  - Each individual survival is $y_{i,j} \sim \text{Bern}(\phi)$ for $j = 1, \ldots, N$
  - $y_i = \sum_{j=1}^{N} y_{i,j}$
  - Conditional independence among N individuals

- Poisson:

  $y_i \sim \text{Pois}(N \cdot \phi)$, for $i = 1, \ldots, n$
  - Each individual survival is $y_{i,j} \sim \text{Pois}(\phi)$ for $j = 1, \ldots, N$
  - $y_i = \sum_{j=1}^{N} y_{i,j}$
  - Conditional independence among N individuals

Data Model and Likelihood

Joint Distribution: $\mathbf{y} \sim [\mathbf{y}|\boldsymbol{\theta}]$

- $\mathbf{y} = (y_1, \ldots, y_n)'$: $n \times 1$ vector of observations
- $\boldsymbol{\theta}$: parameter(s)
- $[\mathbf{y}|\boldsymbol{\theta}]$: joint probability distribution of all $\mathbf{y}$ given $\boldsymbol{\theta}$

Likelihood Function: $L(\boldsymbol{\theta}|\mathbf{y}) = c \cdot [\mathbf{y}|\boldsymbol{\theta}]$

- $L(\boldsymbol{\theta}|\mathbf{y})$: function of parameter(s) given data
- c: ignorable constant (often omitted)
- $[\mathbf{y}|\boldsymbol{\theta}]$: joint PDF (if y cts) or PMF (if y discrete)

Joint Data Model

Joint Distribution for Dependent $\mathbf{y}$:

$$[\mathbf{y}|\boldsymbol{\theta}] = [y_n|y_1, \ldots, y_{n-1}, \boldsymbol{\theta}] \cdots [y_2|y_1, \boldsymbol{\theta}][y_1|\boldsymbol{\theta}]$$

Joint Distribution for Independent $\mathbf{y}$:

$$[\mathbf{y}|\boldsymbol{\theta}] = [y_n|\boldsymbol{\theta}] \cdots [y_2|\boldsymbol{\theta}][y_1|\boldsymbol{\theta}]$$

Likelihood for Independent $\mathbf{y}$:

$$L(\boldsymbol{\theta}|\mathbf{y}) = \prod_{i=1}^{n} [y_i|\boldsymbol{\theta}]$$

Likelihood for Simulated Data

Simulated Data:

$y_i \sim \text{Binom}(N, \phi)$, for $i = 1, \ldots, n$

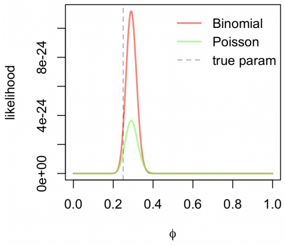
- $N = 10$
- $\phi = 0.25$
- $n = 30$
- $\mathbf{y} = (2, 4, 2, 4, 5, 0, 3, 4, 3, 2, 5, 2, 3, 3, 1, 4, 2, 0, 2, 5, 4, 3, 3, 6, 3, 3, 3, 3, 2, 1)'$

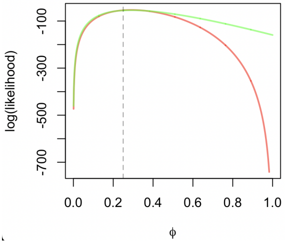
Poisson Likelihood:

$L(\phi|\mathbf{y}) = \prod_{i=1}^{n} \frac{(\phi N)^{y_i} e^{-\phi N}}{y_i!}$

Binomial Likelihood:

$L(\phi|\mathbf{y}) = \prod_{i=1}^{n} \binom{N}{y_i} \phi^{y_i} (1-\phi)^{N-y_i}$

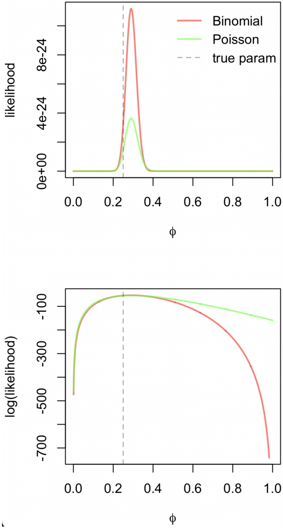




Maximum Likelihood Estimation

Find ϕ that maximizes $L(\phi|\mathbf{y})$:

- Analytical
  1. compute log of likelihood function, drop additive terms w/o ϕ
  2. differentiate log-likelihood wrt ϕ
  3. set equal to zero and solve for ϕ
- Numerical
  - Grid search
  - Optimization algorithm



Binomial MLE

Binomial Likelihood:

$L(\phi|\mathbf{y}) = \prod_{i=1}^{n} \binom{N}{y_i} \phi^{y_i}(1-\phi)^{N-y_i}$

Binomial Log-Likelihood:

$l(\phi|\mathbf{y}) = c + \left(\sum_{i=1}^{n} y_i\right) \log(\phi) + \left(\sum_{i=1}^{n}(N - y_i)\right) \log(1-\phi)$

Objective Function:

$0 = \left(\sum_{i=1}^{n} y_i\right) \frac{d}{d\phi} \log(\phi) + \left(\sum_{i=1}^{n}(N - y_i)\right) \frac{d}{d\phi} \log(1-\phi)$

Binomial MLE:

$\hat{\phi}_{\text{binom}} = \frac{\sum_{i=1}^{n} y_i}{nN}$

Poisson MLE

Poisson Likelihood:

$$L(\phi|\mathbf{y}) = \prod_{i=1}^{n} \frac{(\phi N)^{y_i} e^{-\phi N}}{y_i!}$$

Poisson Log-Likelihood:

$$l(\phi|\mathbf{y}) = c + \left(\sum_{i=1}^{n} y_i\right) \log(\phi N) - nN\phi$$

Objective Function:

$$0 = \left(\sum_{i=1}^{n} y_i\right) \frac{d}{d\phi} \log(\phi) + nN \frac{d}{d\phi} \phi$$

Poisson MLE:

$$\hat{\phi}_{\text{pois}} = \frac{\sum_{i=1}^{n} y_i}{nN}$$

Variance of MLEs

For $n \to \infty$: $\text{Var}(\hat{\phi}) = -1/\text{E}\left(\frac{d^2}{d\phi^2} l(\phi|\mathbf{y})\right)$

Binomial:

$\text{Var}(\hat{\phi}_{\text{binom}}) = \frac{\hat{\phi}(1-\hat{\phi})}{nN}$

Poisson:

$\text{Var}(\hat{\phi}_{\text{pois}}) = \frac{\hat{\phi}}{nN}$

$$\text{Var}(\hat{\phi}_{\text{binom}}) < \text{Var}(\hat{\phi}_{\text{pois}})$$

Asymptotic Confidence Interval: $\hat{\phi} \pm 1.96\sqrt{\text{Var}(\hat{\phi})}$

Inference for Simulated Data

Simulated Data:

$y_i \sim \text{Binom}(N, \phi)$, for $i = 1, \ldots, n$

- $n = 30$
- $\mathbf{y} = (2, 4, 2, 4, 5, 0, 3, 4, 3, 2, 5, 2, 3, 3, 1, 4, 2, 0, 2, 5, 4, 3, 3, 6, 3, 3, 3, 3, 2, 1)'$

Binomial Inference:

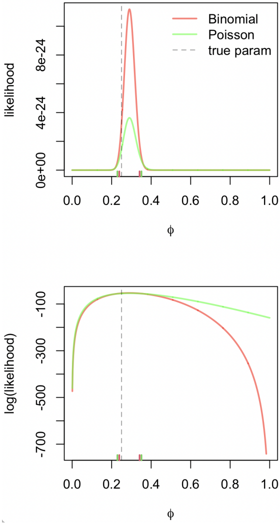
$\hat{\phi}_{\text{binom}} = 0.29$

$\sqrt{\text{Var}(\hat{\phi}_{\text{binom}})} = 0.026$

Poisson Inference:

$\hat{\phi}_{\text{pois}} = 0.29$

$\sqrt{\text{Var}(\hat{\phi}_{\text{pois}})} = 0.031$



Numerical MLE

```
ll.binom <- function(y,N,phi){
  -sum(dbinom(y,N,phi,log=TRUE))
}

phi.hat.opt.binom=optim(.5,ll.binom,method="Brent",
  lower=0,upper=1,N=N,y=y)$par

phi.hat.opt.binom
[1] 0.29
```